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BY

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Cultivated mind is the guardian genius of democracy. . . . . It is the only dictator that freemen acknowledge and the only security that freemen desire.

President Mirabeau B. Lamar.

The benefits of education and of useful knowledge, generally diffused through a community, are essential to the preservation of a free government.

President Sam Houston.

THE ERROR-RISK OF THE MEDIAN COMPARED WITH THAT OF THE ARITHMETIC MEAN.

BY EDWARD L. DODD.

If several leaves * from a tree have been collected, the question arises: What is the typical or representative length of leaf for this tree? Or, if several measurements have been made of the same quantity, the question arises: What shall we accept as an approximation for the "true value" of the unknown magnitude? The "typical value" of the length of a leaf and the "true value" of a magnitude exist usually only as mathematical abstractions. Nevertheless, if we arrange the leaves in the order of their magnitude—as to length—it appears more reasonable to select the middle leaf as the typical leaf than to select the first and shortest leaf. The length of this middle leaf is called the *median* of the lengths. On the other hand, it is not so clear that the median is preferable to the average—the *arithmetic mean*—of the lengths.† In fact, it seems probable that sometimes the arithmetic mean is to be preferred, and sometimes the median. Nevertheless, we may with advantage investigate these two functions of the measurements from the standpoint of general statistical theory.

In this paper, the Gaussian Probability Law will be used as a criterion. It is not pretended that all distributions follow closely this law in its most simple—symmetric—form. But the Gaussian Law has been commonly accepted as representing the ideal distribution, both on account of theoretic considerations and by reason of experimental verifications. For example; an urn contains N balls, of which M are white. The balls are mixed, and n balls are drawn—where $n<N$—and then returned to the urn. Of these, m are found to be white; and thus m/n is a "measurement" of the "true value," M/N, which in this case is *exactly known*. The distribution of the repeated "measurements" about the "true value," M/N can be studied experimentally.

For mathematical treatment, it is assumed that a "true value" or "typical value" or "normal value" exists: let this be denoted by a. The difference, $a-m$, where m is a measurement, is called the *error*‡ of that measurement. The Gaussian Law assumes the existence of a *measure of precision*, h, for a set of measurements

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*See King's Elements of Statistical Method, p. 101, Macmillan (1912).

†For an elementary discussion of the advantages and disadvantages of the arithmetic mean, the median, the mode, etc., see King, loc. cit. Chap. XII.

‡Such terms as "deviation," "departure," are also used to designate $a-m$ or $m-a$.

made under like circumstances; and then states that the probability that the error of a measurement will lie between x_1 and x_2 is

$$p=\frac{h}{\sqrt{\pi}}\int_{x_1}^{x_2} e^{-h^2x^2}\,dx. \tag{1}$$

In the language of "large numbers," this means that when the number, n, of measurements is large, it is to be expected that about pn of these measurements will have errors greater than x_1 and at the same time less than x_2. This law (1) is frequently expressed as follows: The probability that the error of a measurement will lie between x and $x+dx$ is

$$\frac{h}{\sqrt{\pi}}\, e^{-h^2x^2}\,dx. \tag{2}$$

Let

$$\Theta(x)=\frac{2}{\sqrt{\pi}}\int_{0}^{x} e^{-t^2}\,dt. \tag{3}$$

Then from (1) the probability that the error of a measurement will lie between $-x$ and x is $\Theta\,(hx)$.

As preliminary to the treatment of the median, let us find the probabilty that: The first error—the error of the first measurement—will lie between x and $x+dx$; the second error will be less than the first error; and the third error will be greater than the first error. The probability of the first part of this compound event is given by (2). For the moment, suppose that x is positive, and that the first error is exactly x. Then the probability that the second error will be less than the first error,— i. e., that the second error will lie between $-\infty$ and x,—is $\frac{1}{2}+\frac{1}{2}\,\Theta\,(hx)$. The probability that the third error will be greater than the first error is $\frac{1}{2}-\frac{1}{2}\,\Theta\,(hx)$. The probability of the compound event is then

$$\frac{1}{4}\,\frac{h}{\sqrt{\pi}}[1-\Theta^2(hx)]\,e^{-h^2x^2}\,dx; \tag{4}$$

that is, the product of the probabilities of the three components.

The restriction that x be positive is not essential; since $\Theta\,(-x)=-\Theta\,(x)$. The supposition that the first error is *exactly* x is a common device, incident to the extension of the law for compound probability—deduced from the consideration of a *finite* number of possibilities—to the *continuum* with its *infinity* of real numbers. We do not deny that such an extension involves some axiom.

If now $2n+1$ measurements are to be made, the probability that: The first error will lie between x and $x+dx$; the next n errors will be (algebraically) less than the first error; and the last n errors will be greater than the first error, is obtained from (4) by raising $\frac{1}{4}$ and the bracket each to the *nth* power. If, however, we simply require that *some unspecified one* of the $2n+1$ errors shall lie between x and $x+dx$, and that of the $2n$ remaining errors just n errors shall be less than x, the coefficient $(2n+1)\frac{(2n)!}{(n!)^2}$ is also needed.

Suppose now that the error of the median of $2n+1$ measurements is x. Then the errors of n of the measurements must be (algebraically) less than or equal to x; and the errors of the remaining n measurements must be greater than or equal to x. The probability that the error of the median of $2n+1$ measurements will lie between x_1 and x_2 is, then,

$$\frac{(2n+1)!}{4^n(n!)^2}\frac{h}{\sqrt{\pi}}\int_{x_1}^{x_2}[1-\Theta^2(hx)]^n e^{-h^2x^2}dx. \tag{5}$$

If we set

$$N=\frac{(2n+1)!}{4^n(n!)^2};\ t=hx;\ t_1=hx_1;\ t_2=hx_2; \tag{6}$$

this becomes

$$\frac{N}{\sqrt{\pi}}\int_{t_1}^{t_2}[1-\Theta^2(t)]^n e^{-t^2}dt. \tag{7}$$

An interesting verification of (5) and (7) consists in setting $x_1=-\infty$, $x_2=+\infty$; and introducing the change of variable,

$$u=\Theta(t),\ du=\frac{2}{\sqrt{\pi}}e^{-t^2}dt.$$

Then (7) becomes

$$N\int_0^1(1-u^2)^n du. \tag{8}$$

By the substitution, $u=\sin\phi$, this can be shown† to be equal to unity, the symbol for certainty.

We shall now make use of the conception of the *probable value.* To illustrate: If a gambler is to receive one dollar if he throws an

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†Peirce—A Short Table of Integrals, p. 62.

ace with a die, two dollars if he throws a two-spot, etc., his *expectation* on a single throw is defined as

$$\tfrac{1}{6}(1)+\tfrac{1}{6}(2)+\tfrac{1}{6}(3)+\tfrac{1}{6}(4)+\tfrac{1}{6}(5)+\tfrac{1}{6}(6)=3\tfrac{1}{2}$$

dollars; and this is also called the *probable value* of his intake. It is obtained by multiplying each possible prize by its probability—in this case, $\frac{1}{6}$—and taking the sum of the products. It may be roughly described as the expected average intake. The gambler's "expectation" in sixty throws is $60\times 3\frac{1}{2}=210$ dollars; and this is exactly what he will receive if he throws an ace ten times, and each other spot just ten times.

Now let the absolute or positive value of an error, x, be called the *absolute error*, and be denoted by $|x|$. Then, in accordance with the usual generalization, the *probable value* of the *absolute error* of the *median* is obtained by multiplying the integrand in (5) by $|x|$, and taking $x_1=-\infty$, $x_2=\infty$; or, what is equivalent, multiplying the integrand by $2x$, and taking $x_1=0$, $x_2=\infty$.

We can obtain directly a theorem comparing the probable value of the absolute error of the median with that of the arithmetic mean; but we shall obtain it as a corollary to a theorem on error-risk.

In dealing with *error-risk*—"*Fehlerrisiko*"—Czuber* introduces a risk-function with the following characteristics:

$$\mathrm{v}(-\mathrm{x})=\mathrm{v}(\mathrm{x});\ \text{and } \mathrm{v}(\mathrm{x}')>\mathrm{v}(\mathrm{x}) \text{ if } |\mathrm{x}'|>|\mathrm{x}|. \tag{9}$$

It is to be supposed also that v is not negative, and that $v(x)e^{-h^2x^2}$ is integrable from 0 to ∞. The most simple risk-functions are, perhaps,

$$|x|,\ x^2,\ |x^3|,\ x^4, \text{ etc.}$$

By the *error-risk* is meant the *probable value* of $v(x)$, where x is the error of a measurement or of a function of the measurements. Hence, from (5) and (6), the *error-risk* of the *median* of $2n+1$ measurements is

$$\mathrm{P}_1=\frac{2\mathrm{N}}{\sqrt{\pi}}\int_0^\infty \mathrm{v}\left(\frac{\mathrm{x}}{\mathrm{h}}\right)[1-\Theta^2(\mathrm{x})]^{\mathrm{n}}\,\mathrm{e}^{-\mathrm{x}^2}\mathrm{dx}. \tag{10}$$

This is obtained by first making the substitution, $hx=t$; and then, $t=x$.

Now, the arithmetic mean of $2n+1$ measurements, each subject

*Wahrscheinlichkeitsrechnung I. p. 267.

to the Gaussian Law (1) with measure of precision, h, is itself† subject to the Gaussian Law with measure of precision,

$$H=h\sqrt{2n+1}. \tag{11}$$

Hence, the error-risk of the arithmetic mean of $2n+1$ measurements is

$$P=\frac{2\sqrt{2n+1}}{\sqrt{\pi}}\int_0^{\infty} v\left(\frac{x}{h}\right)e^{-(2n+1)x^2}dx. \tag{12}$$

It will now be shown that P is less than P_1 for each (positive integral) value of n. In the technical sense, as used here, the "*risk*" of *error* in accepting the *arithmetic mean* is, then, *less* than in accepting the *median*. In other words, it is to be expected that the error of the ărithmetic mean will be numerically less than that of the median in Gaussian distributions.

To prove this, consider the two curves, suggested by (12) and (10), viz.,

$$y=F(x)=\frac{2\sqrt{2n+1}}{\sqrt{\pi}}e^{-(2n+1)x^2}, \tag{13}$$

$$y=f(x)=\frac{2N}{\sqrt{\pi}}[1-\Theta^2(x)]^n e^{-x^2} \tag{14}$$

In what follows, we shall consider only values of $x>0$ or $x=0$. We shall first prove that $F(0)>f(0)$. Stirling's formula, in closed form*, is

$$n!=\sqrt{2\pi n}\,n^n e^{-n+\frac{\theta}{12n}},\quad 0<\theta<1. \tag{15}$$

Then, from (6),

$$N=\frac{(2n+1)(2n!)}{4^n(n!)^2}=\frac{(2n+1)(\sqrt{4\pi n}(2n)^{2n}e^{-2n+\frac{\theta'}{24n}}}{4^n(2\pi n)\,n^{2n}e^{-2n+\frac{\theta}{6n}}}$$

$$=\frac{2n+1}{\sqrt{\pi n}}e^{\frac{\theta'}{24n}-\frac{\theta}{6n}}$$

From this, it follows that

$$e^{-\frac{1}{6n}}\sqrt{\frac{2}{\pi}}<\frac{N}{\sqrt{2n+1}}<\sqrt{\frac{2n+1}{\pi n}}\,e^{\frac{1}{24n}}$$

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†Czuber, loc. cit. p. 262, gives a more general statement of this principle.

*Cesaro—Corso di analisi algebrica, pp. 270, 480.

And hence, when $n > 1$,

$$\tfrac{1}{2} < \frac{N}{\sqrt{2n+1}} < 1, \tag{16}$$

which is, indeed, also satisfied when $n=1$. Hence, $F(0)>f(0)$; since $\Theta(0)=0$.

We shall next prove that $F(x)<f(x)$ when $x>2$. In other words, beyond $x=2$ the curve, $y=F(x)$, lies wholly below the curve, $y=f(x)$. Since $\Theta(\infty)=1$,

$$1-\Theta(x)=\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-x^2}dx.$$

Now, if $x>2$,

$$8xe^{-x^2}<1;$$

and thus,

$$2\int_x^\infty \left(4xe^{-x^2}\right) e^{-x^2}dx<\int_x^\infty e^{-x^2}dx.$$

But the left member is equal to $2e^{-2x^2}$; and hence,

$$1-\Theta(x)>2e^{-2x^2};$$

$$[1-\Theta^2(x)]^n>2^n e^{-2nx^2}$$

This, with (16), proves that $F(x)<f(x)$ when $x>2$.

Hence, the curves (13) and (14) must intersect at least once—as stated before, we are not considering negative values of x— and, if they intersect twice, they must intersect at least three times. By the indirect method, we shall prove that the curves cannot intersect in three points; and hence, they intersect but once. Suppose, then, that there are three points of intersection. And let

$$D(x)=F(x)-f(x);\ c^n=\frac{N}{\sqrt{2n+1}},\ c>0. \tag{17}$$

Then

$$D(x)=\frac{2}{\sqrt{\pi}}e^{-x^2}\sqrt{2n+1}\left\{e^{-2nx^2}-c^n[1-\Theta^2(x)]^n\right\}$$

One factor of $D(x)$ is

$$g(x)=e^{-2x^2}-c[1-\Theta^2(x)];$$

and the product of the remaining factors is positive. Then, since $D(x)$ has three positive roots, $g(x)$ also has three positive roots; and

$$g'(x) = -4xe^{-2x^2} + 2c\,\Theta(x)\frac{2}{\sqrt{\pi}}e^{-x^2}$$

has two positive roots, by Rolle's Theorem.
Thus,

$$G(x) = -xe^{-x^2} + \frac{c}{\sqrt{\pi}}\,\Theta(x)$$

has two positive roots. But $G(0)$ is also zero. And hence

$$G'(x) = (2x^2 - 1)e^{-x^2} + \frac{2c}{\pi}e^{-x^2}$$

has two positive roots. But

$$2x^2 - 1 + \frac{2c}{\pi}$$

can not be zero for more than one positive value of x. Thus, the supposition that (13) and (14) intersect in more than one (positive) point, leads to an impossibility.

Let b be the abscissa of the point of intersection, and let A be the area between the two curves (and the Y axis) from $x=0$ to $x=b$. Then, the area between the two curves from $x=b$ to $x=\infty$ is also A; since the total area beneath each curve is unity—see (8).

Now, from (10), (12), (13), (14), (17),

$$P = \int_0^\infty F(x)v\left(\frac{x}{h}\right)dx = \int_0^b D(x)v\left(\frac{x}{h}\right)dx + \int_0^b f(x)v\left(\frac{x}{h}\right)dx + \int_b^\infty F(x)v\left(\frac{x}{h}\right)dx; \tag{18}$$

$$P_1 = \int_0^\infty f(x)v\left(\frac{x}{h}\right)dx = \int_0^b f(x)v\left(\frac{x}{h}\right)dx + \int_b^\infty F(x)v\left(\frac{x}{h}\right)dx + \int_b^\infty -D(x)v\left(\frac{x}{h}\right)dx. \tag{19}$$

But the first integral on the right in (18) is less than $Av\left(\frac{b}{h}\right)$; since, by (9), v is an increasing function of x. The last two integrals in (18) are the same as the first two integrals on the right in (19). In the last integral of (19), the function $-D(x)$ is positive*; since $F(x)<f(x)$ when $x>b$. This integral is greater than $Av\left(\frac{b}{h}\right)$ Hence

$$P<P_1. \tag{20}$$

THEOREM

The error-risk of the median of an odd number† of measurements, each subject to the Gaussian Law (1), *is greater than the error-risk of the arithmetic mean of these measurements.*

COROLLARY

The probable value of the absolute error of the median of an odd number† of measurements, each subject to the Gaussian Law (1), *is greater than that of the arithmetic mean of these measurements.*

This corollary is also evident from mechanical considerations,—when it has been proved that $F(0)>f(0)$; and that the curves, $y=F(x)$ and $y=f(x)$, intersect but once. For the two probable values, P and P_1 are simply the abscissas of the centers of gravity of the areas beneath these two curves, respectively.

Now a center of gravity problem may be looked upon as a problem of finding an *average*,—involving, perhaps, a passage to a limit. And there is enough resemblance between a *probable value* and an *average* to lead to a suspicion that a method of comparison based upon probable value might favor the arithmetic mean, M, above all other functions of the measurements. That this is not indeed the case would appear from the fact that *there are functions*** with an *error-risk smaller* than that of the *arithmetic mean*. And futhermore, these comparisons based upon probable value are in harmony—so far as they go—with comparisons obtained by another method††.

The median is not one of the functions considered by Czuber‡ in his treatment of error-risk. For, although the median is a continuous function of its arguments—the measurements—the first partial derivarives do not always exist; and thus there are

*It might be zero at just one point, so far as has been proved; but this does not affect the argument.

†The degenerate case, $n=1$, is not here considered.

**As found in a former investigation. See Monatshefte fuer Mathematik und Physik, 1913, p. 270. For example: $\left(1-\frac{1}{n^2}\right)M$, when n is large enough,—using (8) p. 270.

††Annals of Mathematics, June 1913, pp. 186-198.

‡Loc. cit. p. 275.

points about which the median can not be given a Taylor's development. To illustrate: Suppose the measurements are x, y, and z; and that the median *is $F(x, y, z)$; and suppose $x<z$. If, now, $x<y<z$, then the partial derivative, $F_y'(x, y, z)=1$. But, if $y=x$, the forward F_y' is equal to 1; whereas, the backward F_y' is equal to 0. And thus, at this point, F_y' itself does not exist. Czuber considers functions that can be given a Taylor's development.

The attempt has been made to show that the Gaussian Probability Law* is a logical consequence of the "Principle," that the arithmetic mean is the "most probable value" of the unknown true value. But Bertrand† gives an example to show that the Law and "Principle" are not strictly compatible. What *function* of the measurements the *Gaussian Law endorses* above all other functions is *not at present known.* And it *may* well *be* that there is *no such function.*

The theorem of this paper does not recommend the use of the arithmetic mean in preference to the median under all circumstances; but merely when it is assumed that the distribution is a Gaussian distribution.

But, in view of this theorem, if there is any reason for supposing —in a particular case—that the median gives a more satisfactory result than the arithmetic mean, this is also a reason for supposing that the distribution is not Gaussian. And thus, any formula for "probable error" or like expression deduced from the assumption of the Gaussian Law should be viewed with considerable suspicion, so far as its application to the particular case is concerned.

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*This F is entirely distinct from that used in (13).

†Calcul des Probabilites (1889) p. 180.